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Measurement unit invariant coefficients in multiplicative-logarithmic functions

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It has been demonstrated that the coefficients of the non-multiplicative terms of a multiplicative-logarithmic function are dependent on the units used. In addition, Dickey–Fuller tests on the multiplicative terms are unit sensitive. It is demonstrated that a simple transformation circumvents these problems It is still true that individual coefficients of logarithmic variables in such functions are meaningless and that the Dickey–Fuller test is disturbingly arbitary. However, the transformation appears to increase the meaningfulness of a technical change trend in a translog cost function.

I. INTRODUCTION

Hunt and Lynk (1993) demonstrate that the coefficients of the non-multiplicative terms of a multiplicative-logarithmic function are dependent on the units used Ågren and Jonsson (1992) have also pointed out that Dickey–Fuller tests for unit roots in the interactive terms of a multiplicative logarithmic function are affected by the measurement-unit problem.

Both Hunt and Lynk (1993) and Ågren and Jonsson (1992) are correct, but a simple transformation of the data makes the coefficients invariant to the units of measurement. The transformation subtracts a measure of central tendency of the logarithm of each variable from its logarithm in each period. For normally distributed variables the expected value of all measures coincide. Therefore, in a large sample the coefficients will converge on unique unit-invariant values irrespective of the central measure employed, but in general the unit-invariant coefficients do not appear to be any more economically meaningful. However, I show that in the case of a translog function used to estimate the underlying trend in technical change the transformation results in coefficients that have a meaningful interpretation.

II. THE TRANSFORMATION

The aim is to find an operator $M(\,)$ that meets the following conditions:

$$M(\ln(X_{it})) = M(\ln(X_{it}) - \ln(a)) \qquad (1)$$

where a is the base of a change in measurement units, and:

$$M(\ln(X_{it})) - M(\ln(X_{is})) = \ln(X_{it}) - \ln(X_{is}) \quad \forall_{t,s} \qquad (2)$$

which follows from the definition of regression coefficients as the partial derivatives of the dependent variable with respect to the relevant independent variable Rearranging Equation 2:

$$M(\ln(X_{it})) - \ln(X_{it}) = M(\ln(X_{is})) - \ln(X_{is}) \qquad (3)$$

shows that the operator works through subtracting the same constant from all observations of $\ln X_i$ that is:

$$M(\ln(X_{it})) = \ln(X_{it}) - f(\ln(X_{i1}), \ldots, \ln(X_{iT})) \qquad (4)$$

where $f(\ln(X_{i1}), \ldots, \ln(X_{iT}))$ is a constant for all t Applying the change of units to Equation 4 yields:

$$M(\ln(X_{it}) - \ln(a)) = \ln(X_{it}) - \ln(a) - f(\ln(X_{i1}), \ldots, \ln(X_{iT}) + f(\ln(a)) \qquad (5)$$

From Equation 1 it is clear that $\ln(a) = f(\ln(a))$. Any measure of central tendency will meet these criteria. For a normally distributed variable the expected values of all such functions coincide. Therefore, as the sample size increases the coefficients will converge on unique values irrespective of the measure employed. The simplest such transformation is to subtract the mean. No transformation of the dependent variable is necessary as this affects only the constant term, as Hunt and Lynk (1993) state

This approach is not simply a regression using the deviations from the means of the variables. Such an equation

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would always transform the dependent variable, have no constant, and the multiplicative terms would be in the form of $M(\ln(X_{it})\ln(X_{jt}))$ rather than $M(\ln(X_{it}))\,M(\ln(X_{jt}))$. Incidentally, the first difference operator also results in unit invariant coefficients. However, applying this operator to a multiplicative-logarithmic function changes all the coefficients in the regression, not just the coefficients in the non-multiplicative terms, and dramatically alters the statistical properties of the model.

Does this transformation also solve the problem noted by Ågren and Jonsson (1992)? The Dickey–Fuller test for integration of an interactive term is a t-test on β in the following regression:

$$\Delta(Z_t(\ln(X_t)) = \alpha + \beta Z_{t-1}\ln(X_{t-1}) \qquad (6)$$

where Z may be logarithmic or non-logarithmic. If β is insignificantly less than zero or greater than zero then $Z_t\ln(X_t)$ is integrated of at least order one. Changing the units of measurement of X results in:

$$Z_t\ln(X_t/a) - Z_{t-1}\ln(X_{t-1}/a) = \alpha + \beta Z_{t-1}\ln(X_{t-1}/a) \qquad (7)$$

Therefore applying the transformation both before and after the change of units will clearly result in a unit-invariant regression.

III. DEMONSTRATION

I use the same translog production function as Hunt and Lynk (1993) but use US macro data for 1948 to 1990 as a matter of convenience (see Stern, 1993, for details). In the first model, I indexed all variables to 1 in 1948 before taking logarithms and estimating the untransformed model. Then I subtracted the mean from all the logarithmic variables and the time trend, and estimated the transformed model. In the second model all the variables were indexed to 100 in 1948, and in the third model only capital was indexed to 100, the other variables being indexed to 1. Table 1 presents the results for each of the three models with the transformation and without the transformation These clearly demonstrate that the transformation results in a model whose coefficients and standard errors are invariant to the units of measurement. Note that unless the transformation is also applied to the trend variable and to the dependent variable, as I did in these examples, the constant term will be dependent on the units of measurement The table also presents the Dickey–Fuller test for the integration of the interaction term. As Ågren and Jonsson (1992) state, the test is sensitive to the units of measurement used, but the transformation results in a test that is independent of the units of measurement.

IV DISCUSSION

Does this transformation produce a meaningful set of coefficients or simply another set of, albeit unit-invariant, essentially meaningless coefficients? First, I examine the cases referred to by Hunt and Lynk (1993) and the results of Ågren and Jonsson (1992) and then review the implications

Table 1. *OLS estimates of a translog production function for the US macroeconomy 1948–90*[a]

	I		II		III	
	Untrans-formed	Trans-formed	Untrans-formed	Trans formed	Untrans-formed	Trans-formed
Constant	−0.0081	0.0411	−16.4968	0.0411	−11.4034	0.0411
	(−1.2061)	(13.2334)	(−1.2153)	(13.2334)	(−1.6624)	(13.2334)
ln*K*	0.2457	−0.0668	−1.8450	−0.0668	4.7032	−0.0668
	(2.5051)	(−0.9713)	(−0.4363)	(−0.9713)	(1.5820)	(−0.9713)
ln*L*	1.0295	1.0886	9.7307	1.0886	−5.5187	1.0886
	(8.3978)	(16.4277)	(0.9616)	(16.4277)	(−0.7683)	(16.4277)
$(\ln K)^2$	−0.4840	−0.4840	−0.4840	−0.4840	−0.4840	−0.4840
	(−1.5013)	(−1.5013)	(−1.5013)	(−1.5013)	(−1.5013)	(−1.5013)
$(\ln L)^2$	−1.6557	−1.6557	−1.6557	−1.6557	−1.6557	−1.6557
	(−0.8846)	(−0.8846)	(−0.8846)	(−0.8846)	(−0.8846)	(−0.8846)
ln*K*ln*L*	1.4219	1.4219	1.4219	1.4219	1.4219	1.4219
	(0.9143)	(0.9143)	(0.9143)	(0.9143)	(0.9143)	(0.9143)
t	0.0156	0.0156	0.0156	0.0156	0.0156	−0.0156
	(5.9206)	(5.9206)	(5.9206)	(5.9206)	(5.9206)	(5.9206)
DF test on ln*K*ln*L*	3.7619	−0.9524	−0.0794	−0.9524	0.6401	−0.9524

[a] I untransformed: labour (L), capital (K), dependent variable GDP, all indexed to 1948 = 1 I transformed· indexed to 1948 = 1 and mean subtracted from the log of each variable. II untransformed: all variables indexed to 100 in 1948 II transformed· indexed to 1948 = 100 and mean subtracted from the log of each variable. III untransformed. only capital indexed to 100 in 1948 others indexed to 1 in 1948. III transformed: only capital indexed to 100 in 1948 others indexed to 1 in 1948 and mean subtracted from the log of each variable. t-statistics in parentheses.

of the units problem for measuring technical change using translog functions. Blanchflower and Oswald (1990) estimate the following equation:

$$W_t = \alpha_0 + \alpha_1 \ln(U_t) + \alpha_2 [\ln(U_t)]^3 + \sum_{i=1}^{11} \beta_i Z_{it} \quad (8)$$

where U is the level of unemployment, W the real wage rate, and the Z_i various control variables. Hunt and Lynk (1993) criticize Blanchflower and Oswald's clear reference to the significance of α_1 and α_2: 'the unemployment rates become insignificant' (p. 227) Blanchflower and Oswald (1990) should have carried out a joint test on the significance of α_1 and α_2 in order to test whether the unemployment rates are insignificant. No specific economic meaning can be attached to any of the α_i with or without correction of the units problem (which is not the case of the β_i) α_1 and α_2 are not partial derivatives in an economic sense as a change in $\ln(U_t)$ affects the value of the multiplicative term as well. α_0 is the level of W when the sum of all the other terms is zero. Adjustment of the logarithmic variables does not seem to make this latter condition any more meaningful.

Evans and Heckman (1984; 1986) used a standard translog cost function, stating· 'The discrepancy in the base used for the cost and input prices thus only affects the estimated intercept of the cost function because we adopt a logarithm specification for the cost function' (1986, p. 856). As Hunt and Lynk (1993) note, this is incorrect Though the transformation would yield measurement-unit-invariant coefficients, this does not result in economically meaningful coefficients, because again the coefficients are partial derivatives in the statistical model but not in the economic model.

Ågren and Jonsson (1992) provide an example where an economic inference can be made on a single parameter afflicted by the measurement units problem Again the transformation yields measurement-unit-invariant coefficients, but they are not apparently more economically meaningful This is particularly disturbing as an individual coefficient – the slope coefficient in the Dickey–Fuller auxiliary regression – is taken to have economic meaning. For example, if the interaction term is integrated of a higher order than the other variables in the regression then the regression cannot represent a cointegrated relationship (Cuthbertson *et al.*, 1992). Generally, the presence or absence of cointegrating relationships is taken to have economic meaning in terms of long-run dynamics.

My final example demonstrates a case where there may be an increase in economic meaningfulness as a result of applying the transformation. In this case, inferences are made on non-logarithmic variables in regression whose other independent variables are logarithmic. The translog cost function is given by:

$$\ln C_t = \beta_0 + \sum_i \beta_i \ln(Z_{it}) + \frac{1}{2}\sum_i \sum_j \beta_{ij} \ln(Z_{it}) \ln(Z_{jt}) \quad (9)$$

$$Z_t = [Q_t, \exp(\tau_t), P_{1t}, \ldots, P_{nt}]'$$

where C is costs, Q is the level of output, τ is the time trend, and $P_1, \ldots, P_n$ are the prices of the n factors of production. The rate of growth of total factor productivity is defined as the partial derivative of the logarithm of costs with respect to time in Equation 9 (Slade, 1989):

$$\frac{\partial C_t}{\partial t} = \beta_\tau + \beta_{\tau\tau}\tau_t + \beta_{\tau Q} \ln Q_t + \sum_i \beta_{\tau P_i} \ln P_{it} \quad (10)$$

The true trend in technology is estimated by $\beta_\tau + \beta_{\tau\tau}\tau_t$. A test for improving technology can be executed by testing whether the mean of $\beta_\tau + \beta_{\tau\tau}\tau_t$ is greater than zero. It is therefore crucial that the other variables are entered into the equation in such a way that the coefficients are invariant to the units of measurement and that the coefficients are meaningful in terms of a technical progress trend. In this case, if the transformation is applied to the independent variables then $\beta_\tau + \beta_{\tau\tau}\tau_t$ is the expected value of $\partial C_t/\partial t$ when all the other independent variables are at their sample mean. This is an intuitively meaningful definition of the trend in underlying technical change. Estimating such an equation without carrying out the transformation can lead to bizarre results with estimated technical change strongly positive or negative depending on the units used In practice Equation 10 will be estimated in place of 9 or jointly with 9 and/or other equations for reasons of multicollinearity. This does not affect the need to apply the transformation

V. CONCLUSIONS

Though Hunt and Lynk (1993) and Ågren and Jonsson (1992) raise important points in their papers, there is a simple method that econometricians can use to circumvent the problem of the dependence on the units of measurement of non-multiplicative coefficients in multiplicative-logarith mic models. It is less clear that this transformation increases the economic meaningfulness of the coefficients. In any case, as Hunt and Lynk (1993) make clear, no economic significance should be attached to individual coefficients in the examples they examine. In Ågren and Jonsson's (1992) example, economic significance may be attached to an individual coefficient afflicted by unit dependence. Though the transformation removes this unit dependence, the resulting coefficients are not apparently any more meaningful. Therefore Ågren and Jonsson's (1992) questioning of the meaningfulness of the DF test in the context of multiplicative-logarithmic models is still relevant. Finally, in the case of estimating a technical change trend using a translog cost function, the transformation does appear to have an intuitive meaningfulness.

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